



TITLE:

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Ergodic properties of the equilibrium processes associated with infinitely many Markovian particles

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Consider a system of independent identically distributed Markov processes which have an invariant measure  $\lambda$ . It is known that if each process starts from each point of a  $\lambda$ -Poisson point process at time zero, these particles are  $\lambda$ -Poisson distributed at every later time  $t > 0$ . In this paper we are concerned with the ergodic properties of the stationary process obtained from such a system of particles, which is called the equilibrium process. Sinai's ideal gas model is a special example of the equilibrium processes.

Let  $(X, \mathcal{B}_X, \lambda)$  be a  $\sigma$ -finite measure space, and denote by  $\mathcal{K}(X)$  a family of all the counting measures on  $X$ , i.e. each element of  $\mathcal{K}(X)$  is an integer-valued measure with a countable set as its support.  $\mathcal{K}(X)$  is equipped with a  $\sigma$ -field  $\mathcal{G}$  which is generated by  $\{\mathcal{J} \in \mathcal{K}(X) ; \mathcal{J}(A) = n\}$ ,  $n \geq 0$ ,  $A \in \mathcal{B}_X$ . An element  $\mathcal{J}$  of  $\mathcal{K}(X)$  is represented by  $\mathcal{J} = \sum_i \delta_{x_i}$  where  $\delta_x(A) = 1$  if  $x \in A$  and  $\delta_x(A) = 0$  if  $x \notin A$ . Let  $\pi_\lambda$  be a probability measure on  $(\mathcal{K}(X), \mathcal{G})$ .

$\pi_\lambda$  is  $\lambda$ -Poisson point process if it satisfies the following ; for any disjoint system  $A_1, \dots, A_n$  of  $\mathcal{B}_X$  such that  $\lambda(A_i) < +\infty$ ,  $i=1, \dots, n$

$\mathcal{J}(A_1), \dots, \mathcal{J}(A_n)$  are independent random variables on  $(\mathcal{K}(X), \mathcal{G}, \pi_\lambda)$ , and

$$\pi_\lambda\{\mathcal{J} ; \mathcal{J}(A_i) = n\} = \left[ \frac{\lambda(A_i)}{n} \right]^n \exp[-\lambda(A_i)], \quad i=1, \dots, n.$$

Next, we define the equilibrium processes associated with Markovian particles.

Let  $X$  be a locally compact separable Hausdorff space and  $\mathcal{B}_X$  be the topological Borel field of  $X$ . Denote by  $W$  the path space of  $X$ , that is, each element of  $W$  is a  $X$ -valued right continuous function with left limit defined on  $(-\infty, \infty)$ , and define the shift operators  $\{\theta_t\}_{-\infty < t < \infty}$  of  $W$  as usual ;  $(\theta_t f)_s = f_{t+s}$  for each  $f$  of  $W$ .

Put  $S = \mathcal{K}(X)$  and  $\Omega = \mathcal{K}(W)$ . Denote by  $\{\Theta_t\}_{-\infty < t < \infty}$  the shift operators on  $\Omega$  induced by the shift operators  $\{\theta_t\}_{-\infty < t < \infty}$  on  $W$ , i.e.

$$\Theta_t \omega = \sum_i \delta_{\theta_t f_i} \quad \text{if } \omega = \sum_i \delta_{f_i}$$

Define  $S$ -valued process  $\{\xi_t(\omega)\}_{-\infty < t < \infty}$  on  $\Omega$  as follows ;

$$\xi_t(\omega) = \sum_i f_t^i \quad \text{if } \omega = \sum_i \delta_{f_i}$$

Then  $\xi_t(\omega)$  is right continuous in  $t$  in a natural topology.

In our situation a motion of one particle is given as a Markov process on  $X$  and denote by  $\{P_t(x, dy)\}$  its transition probabilities.

#### Assumption

$P_t(x, dy)$  is a conservative Feller Markov process and have a Radon invariant measure  $\lambda$ , that is,  $\{P_t(x, dy)\}$  induces a semi-group of contraction operators  $\{T_t\}$  on  $C_\infty(X)$ , and  $\int T_t f(x) \lambda(dx) = \int f(x) \lambda(dx)$  for every  $f$  of  $C_0(X)$ .

Under this assumption  $\{T_t\}$  is, also, a semi-group of contraction operators on  $L^2(X, \mathcal{B}_X, \lambda)$ .

Lemma There is only one  $\sigma$ -finite measure  $Q$  on  $(W, \mathcal{B}_W)$  such that for  $-\infty < t_1 < t_2 < \dots < t_n < +\infty$  and  $\{A_i\}_{i=1,2,\dots,n}$

$$\begin{aligned} & Q[f; f_{t_1} \in A_1, f_{t_2} \in A_2, \dots, f_{t_n} \in A_n] \\ &= \int_{A_1} \lambda(dx_1) \int_{A_2} P_{t_2-t_1}(x_1, dx_2) \dots \int_{A_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \end{aligned}$$

In Particular  $Q$  is  $\{\theta_t\}$ -invariant.

Denote by  $\mathbb{B}$  the  $\sigma$ -field generated by  $\{\omega \in \Omega; \omega(A)=n\} \quad n \geq 0, A \in \mathcal{B}_X$  and put  $\mathbb{P} = \Pi_Q$  (  $Q$ -Poisson point process ). We consider  $(\Omega, \mathbb{B}, \mathbb{P})$  as our basic probability space.

Proposition 1.  $\{\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty}\}$  is a right-continuous Markov stationary process with  $\Pi_\lambda$  as its absolute law.

The Markov process defined above is called the equilibrium process associated with  $[\{T_t\}, \lambda]$ . Our purpose is to investigate the ergodic properties.

Proposition 2. The following (i), (ii), and (iii) are equivalent.

- (i)  $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\})$  is metrically transitive.
- (ii)  $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_K P_s(x, K) \lambda(dx) ds = 0$  for every compact subset  $K$  of  $X$ .
- (iii)  $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (T_s f, g)_{L^2(\lambda)} ds = 0$  for all  $f$  and  $g$  of  $L^2(X, \lambda)$ .

Proposition 3. The following three statements are equivalent.

- (i)  $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty})$  has the strong mixing property.
- (ii)  $\lim_{t \rightarrow \infty} \int_K \lambda(dx) P_t(x, K) = 0$  for all  $f$  and  $g$  of  $L^2(X, \lambda)$ .
- (iii)  $\lim_{t \rightarrow \infty} (T_t f, g)_{L^2(\lambda)} = 0$  for all  $f$  and  $g$  of  $L^2(X, \lambda)$ .

Proposition 4. The following three statements are equivalent.

- (i)  $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty})$  is purely non-deterministic.
- (ii)  $\lim_{t \rightarrow \infty} \int_X \lambda(dx) [P_t(x, K)]^2 = 0$  for every compact subset  $K$  of  $X$ .
- (iii)  $\lim_{t \rightarrow \infty} \|T_t f\|_{L^2(\lambda)} = 0$  for every  $f$  of  $L^2(X, \lambda)$ .

Proposition 5.  $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty})$  is purely non-deterministic if and only if  $\mathbb{E}[\xi_t | \xi_0]$  converges to  $\lambda$  vaguely in probability.

Next we study the Bernoulli property of the shift flow  $\{\Theta_t\}_{-\infty < t < \infty}$

It is easy to see that  $\{\Theta_t\}_{-\infty < t < \infty}$  is a flow on the probability space  $(\Omega = \mathcal{K}(W), \mathbb{B}, \mathbb{P} = \Pi_Q)$ .

So, we define the Bernoulli property in the strong sense.

$(\Omega, \mathcal{B}, \mathbb{P}; \{\Theta_t\}_{-\infty < t < \infty})$  is called a Bernoulli flow if there exists a  $\sigma$ -subfield  $\zeta_0$  of  $\mathcal{B}$  and  $\zeta_t = \Theta_t \zeta_0$  satisfies the following conditions ;

- (i)  $\zeta_t \subset \zeta_s$  for any  $t < s$
- (ii)  $\bigcap_t \zeta_t = \{\phi, \Omega\}$  (mod.  $\mathbb{P}$ )
- (iii)  $\bigvee_t \zeta_t = \mathcal{B}$  (mod.  $\mathbb{P}$ )
- (iv) for any  $t < s$  there exists a  $\sigma$ -subfield  $\zeta_t^s$  of  $\mathcal{B}$  such that  $\zeta_s = \zeta_t \vee \zeta_t^s$  and  $\zeta_t \perp \zeta_t^s$ .

The following lemma is a criterion of the Bernoulli property of our shift flow  $\{\Theta_t\}_{-\infty < t < \infty}$

Lemma Suppose that there exists a real measurable function  $\tau(f)$  on the  $\sigma$ -finite measure space  $(W, \mathcal{B}_W, Q)$  such that for almost all  $f$  w.r.t.  $Q$  (a)  $-\infty < \tau(f) < +\infty$

$$(b) \quad \tau(f) = t + \tau(\Theta_t f) \quad \text{for all } t \text{ of } \mathbb{R}^1.$$

Then,  $(\Omega, \mathcal{B}, \mathbb{P}; \{\Theta_t\}_{-\infty < t < \infty})$  is a Bernoulli flow.

We can show the following proposition by appealing to this lemma.

Proposition 6. Suppose that  $\{T_t\}$  is transient in the sense that

$$\int_0^\infty (T_t \varphi, \varphi)_{L^2(\lambda)} dt < +\infty \quad \text{for every } \varphi \text{ of } C_0^+(X).$$

Then,  $(\Omega, \mathcal{B}, \mathbb{P}; \{\Theta_t\}_{-\infty < t < \infty})$  is a Bernoulli flow.

The equilibrium process  $\{\xi_t\}$  induces a factor flow of  $\{\Theta_t\}$ . Since a Bernoulli flow  $\{\Theta_t\}$  in our sense is a Bernoulli flow in the weak sense (i.e. the automorphism  $\{\Theta_t\}$  is Bernoulli for each  $t \neq 0$ ), the shift flow induced by  $\{\xi_t\}$  is also a Bernoulli flow in the weak sense by the theorem of Ornstein.

Finally we can prove a central limit theorem related to the equilibrium process. Denote by  $G\varphi(x) = \int_0^\infty T_t \varphi(x) dt$  if the integral is well-defined.

Proposition 7. Consider any function  $\varphi \in L^2(X, \lambda)$  which satisfies  $(G|\varphi|, |\varphi|)_{L^2(\lambda)} < +\infty$  and  $(G(|\varphi|G|\varphi|), |\varphi|)_{L^2(\lambda)} < +\infty$ . Then, we have

$$\lim_{t \rightarrow \infty} \mathbb{P}[\omega; \alpha < \frac{\int_0^t \langle \varphi, \xi_s \rangle ds - t \cdot \langle \varphi, \lambda \rangle}{\sqrt{2(\varphi, G\varphi)_{L^2(\lambda)} \times t}} < \beta] = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp(-\frac{x^2}{2}) dx \quad \text{for } \alpha < \beta.$$

Let  $\{Q_t(\xi, d\eta)\}$  the transition probability of the equilibrium process defined in Proposition 1.

In general,  $\{Q_t(\xi, d\eta)\}$  has many invariant measures besides  $\lambda$ -Poisson point processes. In this paper we treated only the equilibrium processes with  $\lambda$ -Poisson point processes  $\Pi_\lambda$  as its absolute law. But this is reasonable because of the following proposition.

Proposition 8. Suppose that

$$\lim_{t \rightarrow \infty} \sup_{x \in X} P_t(x, K) = 0 \quad \text{for any compact set } K \subset X.$$

Let  $\Pi(d\xi)$  an invariant probability measure with respect to  $\{Q_t(\xi, d\eta)\}$ . If the stationary process generated by  $\{Q_t(\xi, d\eta)\}, \Pi(d\xi)$  is metrically transitive,  $\Pi = \Pi_\lambda$  for some  $P_t$ -invariant measure  $\lambda$ .

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